

STABLE PERIODIC WAVE INDUCED IN QUADRATIC NONLINEAR MEDIUM OR SATURABLE TYPE KERR MEDIUM

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ABSTRACT: This work deals with the spatial optical solitons in a nonlinear photonic crystal holding a periodical photonic network. We consider a periodic optical structure formed by an area of a nonlinear thin film waveguide. A model for propagation of the optical field in nonlinear photonic crystals is built up using the Laplace transform. We describe propagation in quadratic nonlinear media of periodic waves in Kerr type saturable. The nonlinear wave equation is solved and the spatial optical solitons occurring in this propagation are put into evidence. We calculate the axial corrections for the linear transversal modes.
KEY WORDS: nonlinear photonic crystal, spatial optical solitons, Laplace transform.

1. INTRODUCTION

Nonlinearity permeates our physical world. The evidence for nonlinear behaviors is present in so many aspects of physics, chemistry, biology, economics, etc., that it is not possible to mention them all in here. Among the most striking and aesthetically appealing manifestations of nonlinearity is the propagation of solitons or, more generally, solitary waves. Strictly speaking, solitons differ from solitary waves because of the remarkable property of integrability of the governing models and its consequence. [1]

The history of solitons dates back to 1834, the year in which John Scott Russell observed that a heap of water in a canal propagated undistorted over several kilometers. For this phenomenon to occur the „solitary wave" must have an unusually large amplitude. This means that the medium in which the wave propagates (water, in this case) must behave in a fundamentally different manner of waves of different amplitudes, that is, its behavior is *nonlinear*. [1]

Formation of optical spatial soliton has attracted a lot of interest following the progress on photorefractive solitons, quadratic solitons, solitons in saturable nonlinear media and topological solitons with time dependent coefficients. Investigations of soliton formation, interaction and soliton induced wave guide are of high interest due to their potential applications in all-optical switching, all-optical interconnectors and wave guide applications. Coupled spatial soliton pairs are obtained using two co-propagating beams in nonlinear media and such pairing has always been an intriguing issue among spatial soliton interactions.

The first spatial solitons in Kerr media were suggested in 1960s. However, it became quickly clear that solitons in Kerr media are stable only in (1+1)D systems while in (2+1)D beams undergo catastrophic collapse [2]. The symbol (m+1)D means that the beams can diffract in m dimensions as they propagate in one dimension.

In [3], A.A. Sukhorukov and Y.S. Kivshar studied dynamics of spatial optical solitons in nonlinear photonic crystals. In [4] N. C. Panoiu and all characterized semidiscrete composite solitons in arrays of quadratically nonlinear waveguides. In [5], Y. S. Kivshar and I. Stegeman described the future of spatial optical solitons as the future of optical waveguides. In [6], M. J. Ahlowitz and all, proposed a new model for propagation of optical field into arrays of nonlinear waveguides. The solution of discrete spatial optical solitons in arrays of waveguides was studied by A.R. Boyd and all [7].
In [8], B.J. Eggleton and all have studying experimental the propagation of nonlinear optical stimulation in Bragg network. In this case, spatial optical solitons appear due equality of nonlinearity and dispersion of optical network.
D.N. Christodonlietes and R.I. Joseph studied in [9], a new class of optical solitons – slow Bragg solitons. Y.S. Kivshar and D.K.Comphell studied [10] highly localized nonlinear modes used potential barrier *for* Pic (Peierls-Naharro). They adjudged two types of highly localized modes in a nonlinear discrete network: the first model is situated in centre of network and the second is no centre in network. In [11], J. Hubne, H.M. van Driel and Y.S. Aitchinson studied ultrafast deflection of spatial solitons in $Al_xGa_{1-x}A_s$ slab waveguides. J.W. Hans, B. Y. Soon, M. Scalara, C. Sihilia studied in [12] coupled-mode equations for Kerr media with periodically modulated linear and nonlinear coefficients. S.F. Mingaleev and Y.S. Kirshar described nonlinear transmission and light localization in photonic-crystal waveguides. In [13], G.I. Stegeman, F. Lederer and all studied discrete quadratic solitons. Similar, in [14] the same authors studied arrays of weakly coupled, periodically poled lithium niobate waveguides. In [15], G.I. Stegeman and all studies one dimensional spatial solitary wave due to cascaded second order nonlinearities in planar waveguides.
Optical beams propagating in a linear medium have a natural tendency to broaden in space (diffraction) and time (dispersion). In a Kerr nonlinear medium, the presence of light modifies its refractive index. The refractive index change resembles the intensity profile of the beams forming an optical lens that increases the index in the center of beam while leaving it unchanged in its tails. This phenomenon is called self-focusing. When the self-focusing exactly balances the diffraction (dispersion), the beam propagates without changing its shape. Such self-trapped waves are called solitary waves, but if they also have particle-like behavior then they are called solitons. Scientists and engineers are pragmatic, however, and they are happy to use the word ’soliton’ to describe what appears to be an excitation that is humped, multi-humped, or localized long enough. In optics, if the light wave (soliton) is localized in time, e.g. an optical pulse in a fiber, then it is

called a temporal soliton. On the other hand, if it is confined spatially, e.g. a self-trapped beam in a waveguide, then it is called a spatial soliton. [16]

In this paper we describe propagation in quadratic nonlinear media of periodic waves in Kerr type saturable. The nonlinear wave equation is solved and the spatial optical solitons occurring in this propagation are put into evidence.

2. METHODOLOGY

In saturable Kerr-type medium the propagation of the optical radiation in (1 + 1) dimensions is described by the nonlinear Schrodinger equation for the slowly varying field amplitude $\Phi(\varsigma,\rho)$: [17]

$$2i\frac{\partial\Phi(\varsigma,\rho)}{\partial\varsigma}+\frac{\partial^2\Phi(\varsigma,\rho)}{\partial\rho^2}-2\frac{\Phi(\varsigma,\rho)|\Phi(\varsigma,\rho)|^2}{1+S|\Phi(\varsigma,\rho)|^2}=0 \qquad (1)$$

The transverse ς and the longitudinal ρ coordinates are scaled in terms of the characteristic pulse (beam) width and dispersion (diffraction) length, respectively; S is the saturation parameter; $\sigma = -1$ (+1) stands for focusing (defocusing) media. [17]

$$\begin{aligned}
&\varsigma = \sigma KZ\\
&\rho = \sqrt{\sigma}K\sqrt{X^2+Y^2}\\
&\varphi = arctg(Y/X)\\
&\eta = \rho\sin\varphi = \sqrt{\sigma}K\sqrt{X^2+Y^2}\sin\varphi\\
&\varsigma = \rho\cos\varphi = \sqrt{\sigma}K\sqrt{X^2+Y^2}\cos\varphi\\
&\sigma = \text{focus parameter}\\
&S = \text{saturation parameter}\\
&\Phi(\varsigma,\rho) = \text{field complex amplitude}
\end{aligned} \qquad (2)$$

The simplest periodic stationary solutions of equation (1) have the form:

$$\Phi(\varsigma,\rho) = U(\varsigma,\rho)\,e^{+2ih\varsigma} \qquad (3)$$

where h is the propagation constant.

By replacing the field in such a form into equation (1), one gets:

$$2i\frac{\partial U(\varsigma,\rho)}{\partial\varsigma}+\frac{\partial^2 U(\varsigma,\rho)}{\partial\rho^2}-2hU(\varsigma,\rho)-\frac{2U^3(\varsigma,\rho)}{1+SU^2(\varsigma,\rho)}=0 \qquad (4)$$

We consider an analytic model, which used Laplace transform of equation (3):

$$\left(\hat{\alpha}(U(\varsigma,\rho))=\int_0^{+\infty}U(\varsigma,\rho)e^{-p\rho}d\rho=\tilde{U}(\varsigma,p)\right)$$

And its reverse is:

$$\hat{\alpha}^{-1}\left(\tilde{U}(\varsigma,p)\right)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\tilde{W}(\varsigma,p)e^{+p\rho}dp=U(\varsigma,\rho) \qquad (5)$$

Equation (4) from Laplace transform takes the form:

$$\frac{\partial\tilde{U}(\varsigma,p)}{\partial i\varsigma}-\left(\frac{p^2}{2}-2h\right)\tilde{U}(\varsigma,p)+\left[\frac{p}{2}U(\varsigma,0)+\frac{1}{2}\left(\frac{\partial U(\varsigma,\rho)}{\partial\rho}\right)_{\rho=0}\right]+\int_0^{+\infty}\frac{U(\varsigma,\rho)}{1+SU^2(\varsigma,\rho)}e^{-p\rho}d\rho=0 \qquad (6)$$

We solve nonlinear equation (6) then we perform the Laplace transform of equation (6).

Nonlinear solutions are obtained as:

$$\begin{aligned}
&iU(\varsigma,\rho)-\frac{1}{\sqrt{2\pi i\varsigma}}e^{-\frac{\rho^2}{2i\varsigma}-2ih\varsigma}+\frac{1}{2}\int_0^{\varsigma}\frac{U(\varsigma',0)}{i(\varsigma'-\varsigma)}\frac{\rho}{2\pi i(\varsigma'-\varsigma)}e^{-\frac{\rho^2}{2i(\varsigma'-\varsigma)}+2ih(\varsigma'-\varsigma)}d(i\varsigma')\\
&+\frac{1}{2}\int_0^{\varsigma}\left(\frac{\partial U(\varsigma',\rho)}{\partial\rho}\right)_{\rho=0}\frac{1}{\sqrt{2\pi i(\varsigma'-\varsigma)}}e^{-\frac{\rho^2}{2i(\varsigma'-\varsigma)}+2ih(\varsigma'-\varsigma)}d(i\varsigma')+\\
&+\int_0^{\varsigma}\int_0^{\infty}d\rho'\frac{U^3(\varsigma',\rho)}{1+SU^2(\varsigma',\rho)}\frac{1}{\sqrt{2\pi i(\varsigma'-\varsigma)}}e^{\frac{(\rho-\rho')^2}{2(\varsigma'-\varsigma)}+2ih(\varsigma'-\varsigma)}d(i\varsigma')=0
\end{aligned} \qquad (7)$$

We use these passages to the limit ($\zeta' \to \zeta$):

$$\begin{aligned}
&\lim_{\varsigma'\to\varsigma}\frac{1}{\sqrt{2\pi i(\varsigma'-\varsigma)}}e^{-\frac{\rho^2}{2i(\varsigma'-\varsigma)}}=\delta(\rho)\\
&\rho\lim_{\varsigma'\to\varsigma}\frac{1}{\sqrt{2\pi i(\varsigma'-\varsigma)}}e^{-\frac{\rho^2}{2i(\varsigma'-\varsigma)}}=\rho\delta(\rho)=0
\end{aligned} \qquad (8)$$

In these conditions the nonlinear equation solution takes the form:

$$\begin{aligned}
&iU(\varsigma,\rho)e^{+2ih\varsigma}-\frac{1}{\sqrt{2\pi i\varsigma}}e^{\frac{\rho^2}{2i\varsigma}}+\frac{1}{2}\int_0^{\varsigma}\left(\frac{\partial U(\varsigma',\rho)}{\partial\rho}\right)_{\rho=0}\delta(\rho)e^{+2ih\varsigma'}d(i\varsigma')+\\
&+\int_0^{\varsigma}\int_0^{\infty}d\rho'\frac{U^3(\varsigma',\rho')}{1+SU^2(\varsigma',\rho')}\delta(\rho-\rho')e^{2ih\varsigma'}d(i\varsigma')=0
\end{aligned} \qquad (9)$$

Or:

$$\begin{aligned}
&iU(\varsigma,\rho)e^{+2ih\varsigma}-\frac{1}{\sqrt{2\pi i\varsigma}}e^{\frac{\rho^2}{2i\varsigma}}+\frac{1}{2}\int_0^{\varsigma}\left(\frac{\partial U(\varsigma',0)}{\partial\rho}\right)_{\rho=0}\delta(\rho)d\left(\frac{e^{2ih\varsigma'}}{2h}\right)+\\
&+\int_0^{\varsigma}\frac{U^3(\varsigma',\rho)}{1+SU^2(\varsigma',\rho)}\delta(\rho-\rho')d\left(\frac{e^{2ih\varsigma'}}{2h}\right)=0
\end{aligned} \qquad (10)$$

and:

$$\begin{aligned}
&iU(\varsigma,\rho)e^{+2ih\varsigma}-\frac{1}{\sqrt{2\pi i\varsigma}}e^{\frac{\rho^2}{2i\varsigma}}+\frac{1}{2}\int_0^{\varsigma}\left(\frac{\partial U(\varsigma',0)}{\partial\rho}\right)_{\rho=0}d\left(\frac{e^{2ih\varsigma'}}{2h}\right)+\\
&+\int_0^{\varsigma}\frac{U^3(\varsigma',\rho)}{1+SU^2(\varsigma',\rho)}d\left(\frac{e^{2ih\varsigma'}}{2h}\right)=0
\end{aligned} \qquad (11)$$

We continue to calculate the integral:

$$\begin{aligned}
&I_1\stackrel{def}{=}\frac{1}{2}\int_0^{\varsigma}\left(\frac{\partial U(\varsigma',0)}{\partial\rho}\right)_{\rho=0}d\left(\frac{e^{2ih\varsigma'}}{2h}\right)=\frac{1}{2}\left[\left(\frac{\partial U(\varsigma',0)}{\partial\rho}\right)_{\rho=0}\frac{e^{2ih\varsigma'}}{2h}\right]_0^{\varsigma}=\\
&\left(\frac{\partial U(\varsigma,0)}{2\partial\rho}\right)_{\rho=0}\left(\frac{e^{2ih\varsigma}}{2h}\right)-\left(\frac{\partial U(0,0)}{2\partial\rho}\right)_{\rho=0}\left(\frac{1}{2h}\right)\\
&I_2\stackrel{def}{=}\int_0^{\varsigma}\left(\frac{U^3(\varsigma',\rho)}{1+SU^2(\varsigma',\rho)}\right)d\left(\frac{e^{2ih\varsigma'}}{2h}\right)=\\
&\left[\left(\frac{U^3(\varsigma,\rho)}{1+SU^2(\varsigma,\rho)}\right)\left(\frac{e^{2ih\varsigma'}}{2h}\right)-\left(\frac{U^3(0,\rho)}{1+SU^2(0,\rho)}\right)\left(\frac{1}{2h}\right)\right]
\end{aligned} \qquad (12)$$

The general term solution is:

$$\begin{aligned}
&iU(\varsigma,\rho)e^{+2ih\varsigma}-\frac{1}{\sqrt{2\pi i\varsigma}}e^{\frac{\rho^2}{2i\varsigma}}+\left\{\left(\frac{\partial U(\varsigma,0)}{2\partial\varsigma}\right)_{\rho=0}\left(\frac{e^{2ih\varsigma}}{2h}\right)-\left(\frac{\partial U(0,0)}{2\partial\rho}\right)_{\rho=0}\left(\frac{1}{2h}\right)\right\}\\
&+\left\{\left(\frac{U^3(\varsigma,\rho)}{1+SU^2(\varsigma,\rho)}\right)\left(\frac{e^{2ih\varsigma}}{2h}\right)-\left(\frac{U^3(0,\rho)}{1+SU^2(0,\rho)}\right)\left(\frac{1}{2h}\right)\right\}=0
\end{aligned} \qquad (13)$$

Roughly speaking in zero order approximation [16]:

$$U(\varsigma,\rho)=\frac{1}{i\sqrt{2\pi i\varsigma}}e^{-\frac{\rho^2}{2i\varsigma}-2ih\varsigma}$$

$$U(\varsigma,\rho)=\frac{1}{i\sqrt{2\pi i\varsigma}}e^{-\left[\frac{\rho^2}{2i\varsigma}+2i\varsigma h\right]}$$

$$U(\varsigma,\rho)=\frac{1}{i\sqrt{2\pi i\varsigma}}e^{-\left(\frac{\rho^2-4\varsigma^2 h}{2i\varsigma}\right)}$$
$$U(\varsigma,\rho)=e^{i\left(\frac{\rho^2-4\varsigma^2 h}{2\varsigma}\right)-i\frac{\pi}{2}-i\frac{\pi}{4}}$$
$$U(\varsigma,\rho)=e^{i\left(\frac{\rho^2-4\varsigma^2 h}{2\varsigma}\right)-i\frac{3\pi}{4}}$$
$$|u(\varsigma,\rho)|=1 \tag{14}$$
$$\lim_{\varsigma\to 0}U(\varsigma,\rho)=\left(\frac{1}{2}\right)\delta\left(\rho^2-4\varsigma^2 h\right)$$
$$\rho^2-4\varsigma^2 h=const$$
$$\sigma K^2\left(X^2+Y^2\right)-4h\sigma^2K^2Z^2=const$$
$$\rho=\sqrt{4h\varsigma^2+const}$$

For h < 0: $\left|\lim_{\varsigma\to 0}U(\varsigma,\rho)\right|=\delta\left(\rho^2+4h\varsigma^2\right)$

Nonlinear equation solution is:

$$iU(\varsigma,\rho)=+\frac{1}{\sqrt{2\pi i\varsigma}}e^{\frac{\rho^2}{2i\varsigma}-2ih\varsigma}-\int_0^{\varsigma}[\frac{1}{2}\left(\frac{\partial U(\varsigma',0)}{\partial\rho}\right)_{\rho=0}+\left(\frac{U^3(\varsigma',\rho)}{1+SU^2(\varsigma',\rho)}\right)]d\left(\frac{e^{2ih(\varsigma'-\varsigma)}}{2h}\right) \tag{15}$$

Let it be the first boundary condition:

$$\left(\frac{\partial U(\varsigma,0)}{\partial\rho}\right)_{\rho=0}=0 \qquad \left(\frac{\partial U(0,0)}{\partial\rho}\right)_{\rho=0}=0 \tag{16}$$

With these conditions the equation takes the form:

$$iU(\varsigma,\rho)=+\frac{1}{\sqrt{2\pi i\varsigma}}e^{\frac{\rho^2}{2i\varsigma}-2ih\varsigma}-\left(\frac{U^3(\varsigma,\rho)}{1+SU^2(\varsigma,\rho)}\right)\frac{1}{2h}-\left(\frac{U^3(0,\rho)}{1+SU^2(0,\rho)}\right)\left(\frac{e^{2ih\varsigma}}{2h}\right) \tag{17}$$

Let it be the compact form of nonlinear terms:

$$\left(\frac{U^3(0,\rho)}{1+SU^2(0,\rho)}\right)=\frac{U^2(0,\rho)}{2\sqrt{S}ch\left[\ln\left(\sqrt{S}U(0,\rho)\right)\right]} \tag{18}$$
$$\left(\frac{U^3(\varsigma,\rho)}{1+SU^2(\varsigma,\rho)}\right)=\frac{U^2(\varsigma,\rho)}{2\sqrt{S}ch\left[\ln\left(\sqrt{S}U(\varsigma,\rho)\right)\right]} \tag{19}$$

Second boundary condition is:

$$U(0,\rho)=\frac{1}{\sqrt{\pi W^2}}e^{-\frac{\rho^2}{W^2}-2ih\varsigma} \tag{20}$$

Associated with the initial conditions we choose a period of evolution:

$$\widetilde{U}(\varsigma,\rho)\approx\frac{1}{\sqrt{2\pi^2 i\varsigma W^2}}e^{\frac{\rho^2}{2i\varsigma}-\frac{\rho^2}{W^2}-2ih\varsigma} \tag{21}$$

Having these conditions, the general solution of the nonlinear wave equation takes the following form:

$$iU(\varsigma,\rho)=\frac{1}{\sqrt{2\pi i\varsigma}}e^{-\frac{\rho^2}{2i\varsigma}-2ih\varsigma}-\left[\left(\frac{\widetilde{U}^2(\varsigma,\rho)}{4h\sqrt{S}ch\left[\ln\left(\sqrt{S}\widetilde{U}(\varsigma,\rho)\right)\right]}\right)-\left(\frac{U^2(0,\rho)e^{-2ih\varsigma}}{4h\sqrt{S}ch\left[\ln\left(\sqrt{S}U(0,\rho)\right)\right]}\right)\right] \tag{22}$$

Roughly speaking the first order approximation the solution of equation takes the form:

$$iU(\varsigma,\rho)=+\frac{1}{\sqrt{2\pi i\varsigma}}e^{-\frac{\rho^2}{2i\varsigma}-2ih\varsigma}-\left[\left(\frac{\frac{1}{\pi W^2}e^{-\frac{2\rho^2}{W^2}-2ih\varsigma}}{4h\sqrt{S}ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]}\right)\right] \tag{23}$$

$$\ln\left(\sqrt{S}\widetilde{U}(\varsigma,\rho)\right)=\ln\sqrt{S}-\frac{1}{2}\ln\left(2\pi^2 i\varsigma W^2\right)-\rho^2\left(\frac{1}{2i\varsigma}+\frac{1}{W^2}\right)-2ih\varsigma$$
$$\ln\left(\sqrt{S}\widetilde{U}(0,\rho)\right)=\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\left(\frac{\rho^2}{W^2}\right) \tag{24}$$

$$iU(\varsigma,\rho)=\frac{1}{\sqrt{2\pi i\varsigma}}e^{-\frac{\rho^2}{2i\varsigma}}+\left[\left(\frac{\frac{1}{\pi W^2}e^{-\frac{2\rho^2}{W^2}}}{4h\sqrt{S}ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]}\right)\right]e^{-2ih\varsigma} \tag{25}$$

$$|U(\varsigma,\rho)|=\left|\left\{\frac{1}{\sqrt{2\pi i\varsigma}}e^{\frac{i\rho^2}{2\varsigma}}+\left[\frac{e^{-\frac{2\rho^2}{W^2}}}{\pi W^2 4h\sqrt{S}ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]}\right]\right\}\right| \tag{26}$$

$$U(\varsigma,\rho)=\frac{1}{\sqrt{2\pi i\varsigma}}e^{i(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4})}+\left[\left(\frac{e^{-\frac{2\rho^2}{W^2}}}{(\pi W^2)(4h\sqrt{S})ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]}\right)\right] \tag{27}$$

$$|U(\varsigma,\rho)|=\sqrt{\left\{\frac{\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)}{\sqrt{2\pi\varsigma}}+\frac{e^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\cdot\left(4h\sqrt{S}\right)\cdot ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]}\right\}^2+\left\{\frac{\sin\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)}{\sqrt{2\pi\varsigma}}\right\}^2} \tag{28}$$

$$\varphi(U(\varsigma,\rho))=arctg\left\{\frac{tg\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)}{1+\frac{\sqrt{2\pi\varsigma}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]}}\right\} \tag{29}$$

$$|U(\varsigma,\rho)|=\sqrt{\left[\begin{array}{l}\left(\frac{1}{2\pi\varsigma}\right)+\frac{e^{-\frac{4\rho^2}{W^2}}}{\left(\pi W^2\right)^2\left(4h\sqrt{S}\right)ch^2\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\left(\frac{\rho^2}{W^2}\right)\right]}+\\+2\frac{\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)}{\sqrt{2\pi\varsigma}}\frac{e^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]}\end{array}\right]} \tag{30}$$

$$\lim_{\varsigma\to 0} U(\varsigma,\rho)=\frac{e^{-\frac{4\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]}$$

$$ch^{-1}\left(\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right)=\left(\frac{1}{e}\right)$$

$$ch\left[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}\right]=e$$

$$\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\frac{\rho^2}{W^2}=\ln\left[e\pm\sqrt{e^2-1}\right] \tag{31}$$

$$\rho^2=W^2[\ln\sqrt{S}-\frac{1}{2}\ln\left(\pi W^2\right)-\ln\left(e\pm\sqrt{e^2-1}\right)]$$

$$\rho_{1,2}=W\sqrt{\ln\left(\sqrt{S}\frac{1}{\sqrt{\pi W^2}\left(e\pm\sqrt{e-1}\right)}\right)}$$

$$\rho_{1,2}=W\sqrt{\ln\left\{\sqrt{\frac{S}{\pi W^2}}\frac{1}{\left(e\pm\sqrt{e^2-1}\right)}\right\}}$$

Soliton width is:

$$\Delta\rho=2W\sqrt{\ln\left\{\sqrt{\frac{S}{\pi W^2}}\frac{1}{e\pm\sqrt{e^2-1}}\right\}}$$

$$i\Phi(\varsigma,\rho)=\frac{1}{\sqrt{2\pi\varsigma}}e^{i(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4})}-\frac{e^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)}\cdot$$
$$\left[\left(\frac{\frac{1}{2\pi\varsigma}e^{i\left(\frac{\rho^2}{\varsigma}-\frac{\pi}{2}\right)}}{ch\left[\left\{\ln\left(\sqrt{\frac{S}{(2\pi\varsigma)(\pi W^2)}}\right)-\left(\frac{\rho^2}{W^2}\right)\right\}+i\left\{\frac{\rho^2}{2\varsigma}-h\varsigma-\frac{\pi}{4}\right\}\right]}\right)-\frac{1}{ch\left[\ln\left(\sqrt{\frac{S}{(\pi W^2)}}\right)-\frac{\rho^2}{W^2}\right]}\right] \tag{32}$$

Define the following sizes:

$$\alpha_R=\ln\left(\sqrt{\frac{S}{(2\pi\varsigma)(\pi W^2)}}\right)-\left(\frac{\rho^2}{W^2}\right)$$
$$\alpha_I=\frac{\rho^2}{2\varsigma}-h\varsigma-\frac{\pi}{4} \tag{33}$$
$$\beta_R=\ln\left(\sqrt{\frac{S}{\pi W^2}}\right)-\left(\frac{\rho^2}{W^2}\right)$$

Substituting these values in (28) is obtained:

$$i\Phi(\varsigma,\rho)=\frac{1}{\sqrt{2\pi\varsigma}}e^{i(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4})}-F(\rho)\left[\left(\frac{\frac{1}{2\pi\varsigma}e^{i\left(\frac{\rho^2}{\varsigma}-\frac{\pi}{2}\right)}}{ch[\alpha_R+i\alpha_i]}\right)-\frac{1}{ch[\beta_R]}\right] \tag{34}$$

This can be expressed as equation (35):

$$i\Phi(\varsigma,\rho)=\frac{1}{\sqrt{2\pi\varsigma}}\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)-\frac{e^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)}\cdot$$
$$\left[\frac{\cos\left(\frac{\rho^2}{\varsigma}-\frac{\pi}{2}\right)ch\alpha_R\cos\alpha_I+\sin\left(\frac{\rho^2}{4}-\frac{\pi}{2}\right)sh\alpha_R\sin\alpha_I}{(2\pi\varsigma)\left[ch^2\alpha_R\cos^2\alpha_I+\sin^2\alpha_I sh^2\alpha_R\right]}-\frac{1}{ch\beta_R}\right]+ \tag{35}$$
$$+\frac{1}{\sqrt{2\pi\varsigma}}\sin\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)-\frac{ie^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)}\cdot$$
$$\left[\frac{\sin\left(\frac{\rho^2}{\varsigma}-\frac{\pi}{2}\right)ch\alpha_R\cos\alpha_I-\cos\left(\frac{\rho^2}{4}-\frac{\pi}{2}\right)sh\alpha_R\sin\alpha_I}{(2\pi\varsigma)\left[ch^2\alpha_R\cos^2\alpha_I+\sin^2\alpha_I sh^2\alpha_R\right]}\right]$$

$$\begin{cases}\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)=\cos\left(\frac{\rho^2}{2\varsigma}\right)\frac{1}{\sqrt{2}}+\sin\left(\frac{\rho^2}{2\varsigma}\right)\frac{1}{\sqrt{2}}\\ \sin\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)=\sin\left(\frac{\rho^2}{2\varsigma}\right)\frac{1}{\sqrt{2}}-\cos\left(\frac{\rho^2}{2\varsigma}\right)\frac{1}{\sqrt{2}}\end{cases}$$
$$\Downarrow$$
$$\begin{cases}\sin\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\left\{\sin\left(\frac{\rho^2}{2\varsigma}\right)-\cos\left(\frac{\rho^2}{2\varsigma}\right)\right\}\\ \cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\left\{\sin\left(\frac{\rho^2}{2\varsigma}\right)+\cos\left(\frac{\rho^2}{2\varsigma}\right)\right\}\end{cases} \tag{36}$$
$$\Downarrow$$
$$\begin{cases}\sin\left(\frac{\rho^2}{\varsigma}-\frac{\pi}{2}\right)=\left\{-\cos\left(\frac{\rho^2}{\varsigma}\right)\right\}\\ \cos\left(\frac{\rho^2}{\varsigma}-\frac{\pi}{2}\right)=\left\{\sin\left(\frac{\rho^2}{\varsigma}\right)\right\}\end{cases}$$

Compact solution of nonlinear wave equation has the form (α_I=0):

$$i\Phi(\varsigma,\rho)=\frac{1}{\sqrt{2\pi\varsigma}}\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)-\frac{e^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)}\cdot$$
$$\left[\frac{\sin\left(\frac{\rho^2}{\varsigma}\right)}{(2\pi\varsigma)}\frac{1}{ch\alpha_R}-\frac{1}{ch\beta_R}\right]+\frac{1}{\sqrt{2\pi\varsigma}}\sin\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)- \tag{37}$$
$$-\frac{ie^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)}\cdot\left[\frac{-\cos\left(\frac{\rho^2}{\varsigma}\right)}{(2\pi\varsigma)}\frac{1}{ch\alpha_R}\right]$$

To simplify we use the next notations:

$$A=\left(\frac{1}{2\pi\varsigma}\right)\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)$$

$$B=\frac{e^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)^2\left(4h\sqrt{S}\right)}\left[\frac{\sin\left(\frac{\rho^2}{\varsigma}\right)}{(2\pi\varsigma)}\frac{1}{ch\alpha_R}-\frac{1}{ch\beta_R}\right]$$

$$C=\frac{1}{\sqrt{2\pi\varsigma}}\sin\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)$$

$$D=\frac{e^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)}\left[\frac{-\cos\frac{\rho^2}{\varsigma}}{(2\pi\varsigma)}\frac{1}{ch\alpha_R}\right]$$

With these notations:

$$\left|\Phi(\varsigma,\rho)\right|=\sqrt{(A-B)^2+(C-D)^2} \tag{38}$$

Then $F(\rho)=\dfrac{e^{-\frac{2\rho^2}{W^2}}}{\left(\pi W^2\right)\left(4h\sqrt{S}\right)}$; $\alpha_I=0$

Noting again:

$$A=\left(\frac{1}{2\pi\varsigma}\right)\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)$$

$$B=F(\rho)\cdot\left[\frac{\sin\left(\frac{\rho^2}{\varsigma}\right)}{(2\pi\varsigma)}\frac{1}{ch\alpha_R}-\frac{1}{ch\beta_R}\right]$$

$$C=\frac{1}{\sqrt{2\pi\varsigma}}\sin\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)$$

$$D=F(\rho)\cdot\left[\frac{-\cos\frac{\rho^2}{\varsigma}}{(2\pi\varsigma)}\frac{1}{ch\alpha_R}\right]$$

$$\left|\Phi(\varsigma,\rho)\right|=\sqrt{(A-B)^2+(C-D)^2} \tag{39}$$

$$\begin{aligned}\left|\Phi(\varsigma,\rho)\right|^2&=\left(\frac{1}{2\pi\varsigma}\right)-\\&-\left(\frac{2}{\sqrt{2\pi\varsigma}}\right)\frac{1}{(2\pi\varsigma)}\left(\frac{F(\rho)}{ch\alpha_R}\right)\cdot\sin\left(\frac{\rho^2}{2\varsigma}+\frac{\pi}{4}\right)+\\&+\frac{2}{\sqrt{2\pi\varsigma}}\left(\frac{F(\rho)}{ch\beta_R}\right)\cos\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)+\left(\frac{F^2(\rho)}{ch^2\beta_R}\right)+\\&+\frac{1}{(2\pi\varsigma)^2}\frac{F^2(\rho)}{ch^2\alpha_R}-2\frac{F^2(\rho)}{ch\beta_R ch\alpha_R}\left(\frac{\sin\left(\frac{\rho^2}{\varsigma}\right)}{2\pi\varsigma}\right)\end{aligned} \tag{40}$$

$$\begin{aligned}i\Phi(\varsigma,\rho)&=\frac{1}{\sqrt{2\pi\varsigma}}e^{i\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)}-\\&-F(\rho)\left[\frac{\sin\left(\frac{\rho^2}{\varsigma}\right)}{(2\pi\varsigma)}\frac{1}{ch\alpha_R}-\frac{1}{ch\beta_R}-\frac{i\cos\frac{\rho^2}{2}}{(2\pi\varsigma)}\frac{1}{ch\alpha_R}\right]\end{aligned} \tag{41}$$

$$i^2\Phi(\varsigma,\rho)=\frac{i}{\sqrt{2\pi\varsigma}}e^{i\left(\frac{\rho^2}{2\varsigma}-\frac{\pi}{4}\right)}+F(\rho)\left[\frac{i}{ch\beta_R}+\frac{i^2}{ch\alpha_R}\frac{e^{-\frac{i\rho^2}{\varsigma}}}{(2\pi\varsigma)}\right] \tag{42}$$

For $\Phi(\varsigma,\rho)=U(\rho)e^{2ih\varsigma}$ evolution equation (6) takes the form:

$$\begin{aligned}&-\left(\frac{p^2}{2}-2h\right)\widetilde{U}(p)+\left[\frac{p}{2}U(0,0)+\frac{1}{2}\left(\frac{\partial U(\varsigma,\rho)}{\partial\rho}\right)_{\rho=0}\right]+\\&+\int_0^{+\infty}\frac{U^3(\varsigma,\rho)}{1+SU^2(\varsigma,\rho)}e^{-p\rho}d\rho=0\end{aligned} \tag{43}$$

At limit $\left(\partial U(\rho)/\partial\rho\right)_{\rho=0}=0$:

$$-\left(\frac{p^2}{2}-2h\right)\widetilde{U}(p)+\left[\frac{p}{2}U(0)\right]+\int_0^{+\infty}\frac{U^3(\rho)}{1+SU^2(\rho)}e^{-p\rho}d\rho=0 \tag{44}$$

$$\widetilde{U}(\rho)=\frac{\left(\frac{p}{2}\right)U(0)+\int_0^{\infty}\frac{U^3(\rho)}{1+SU^2(\rho)}e^{-p\rho}d\rho}{\left(\frac{p^2}{2}-2h\right)}\quad\Rightarrow$$

$$\widetilde{U}(\rho)=\frac{pU(0)+2\int_0^{\infty}\frac{U^3(\rho)}{1+SU^2(\rho)}e^{-p\rho}d\rho}{\left(p^2-4h\right)} \tag{45}$$

$$U(\rho)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{pU(0)+2\int_0^{\infty}\frac{U^3(\rho)}{1+SU^2(\rho)}e^{-p\rho}d\rho}{\left(p^2-4h\right)}e^{p\rho}dp \tag{46}$$

$$\begin{aligned}U(\rho)&=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{pU(0)e^{p\rho}}{p^2-\left(2\sqrt{h}\right)^2}dp+\\&+\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{2e^{p\rho}}{p^2-\left(2\sqrt{h}\right)^2}\left\{\int_0^{\infty}\frac{U^3(\rho')}{1+SU^2(\rho')}e^{-p\rho'}d\rho'\right\}dp\end{aligned} \tag{47}$$

We suppose that: U (ρ) = U1 (ρ) + U2 (ρ), where: $U_1(\rho)=U(0)ch\left(2\sqrt{h}\rho\right)$

$$U_2(\rho)=\int_0^{\infty}d\rho'\frac{U^3(\rho')}{1+SU^2(\rho')}\left[\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{2e^{p(\rho-\rho')}}{\left(p-2\sqrt{h}\right)\left(p+2\sqrt{h}\right)}dp\right]_{=I} \tag{48}$$

Consider the following graphic that we have one pole:

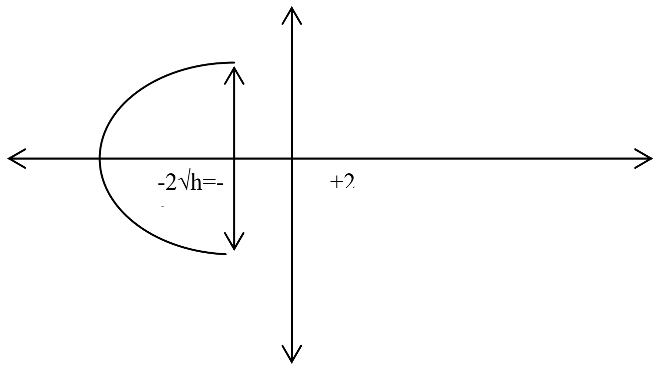

Figure 1. Graphic with pole.

$$U(\rho)=\frac{1}{2\pi i}\int_{\substack{-a-i\infty\\|a|\langle|2\sqrt{h}|}}^{-a+i\infty}\frac{pU(0)+2\int_0^{\infty}\frac{U^3(\rho')}{1+SU^2(\rho')}e^{-p\rho'}d\rho'e^{+p\rho}}{\left(p-2\sqrt{h}\right)\left(p+2\sqrt{h}\right)}dp \tag{49}$$

$$U(\rho)=\frac{1}{2}U(0)-\frac{1}{2\sqrt{h}}\int_0^{\infty}\frac{U^3(\rho')}{1+SU^2(\rho')}e^{-2\sqrt{h}(\rho-\rho')}d\rho' \tag{50}$$

Because we integrated by a single pole: $\frac{1}{2}U(0)\rightarrow U(0)$.

$$U(\rho)=U(0)-\frac{1}{2\sqrt{h}}e^{-2\sqrt{h}\rho}\int_0^{\infty}\frac{U^3(\rho')}{1+SU^2(\rho')}d\left(\frac{e^{2\sqrt{h}\rho'}}{2\sqrt{h}}\right) \tag{51}$$

$$U(\rho)=U(0)-\frac{1}{2\sqrt{h}}e^{-2\sqrt{h}\rho}\cdot$$

$$\cdot\left[\left(\frac{U^3(\rho')}{1+SU^2(\rho')}\right)\frac{e^{+2\sqrt{h}\rho'}}{2\sqrt{h}}\Bigg|_0^\infty-\int_0^\infty\left(\frac{e^{+2\sqrt{h}\rho'}}{2\sqrt{h}}\right)d\left(\frac{U^3(\rho')}{1+SU^2(\rho')}\right)\right] \quad (52)$$

$$U(\rho)=U(0)-\frac{1}{2\sqrt{h}}\left[\frac{e^{-3\left(2\sqrt{h}\rho+\ln 2\sqrt{h}\right)}}{1+e^{-\left(2\sqrt{h}\rho+\ln 2\sqrt{h}\right)}}\right] \quad (53)$$

$$U(\rho)=U(0)-\frac{1}{2\sqrt{h}}\frac{1}{2}\frac{e^{-4\sqrt{h}\rho-\ln\left(2\sqrt{h}\sqrt{S}\right)}}{ch\left[2\sqrt{h}\rho+\ln\left(\frac{2\sqrt{h}}{\sqrt{S}}\right)\right]} \quad (54)$$

$$U(\rho)=U(0)-\frac{1}{4\sqrt{h}2\sqrt{h}\sqrt{S}}\frac{e^{-4\sqrt{h}\rho}}{ch\left[2\sqrt{h}\rho+\ln\left(\frac{2\sqrt{h}}{\sqrt{S}}\right)\right]} \quad (55)$$

$$U(0)=\frac{1}{8h\sqrt{S}}\frac{1}{e^{\ln\left(\frac{2\sqrt{h}}{\sqrt{S}}\right)}+e^{-\ln\left(\frac{2\sqrt{h}}{\sqrt{S}}\right)}}=\frac{1}{\left(4\sqrt{h}\right)\left(4h+S\right)} \quad (56)$$

Nonlinear solutions are obtained as:

$$U(\rho)=\frac{1}{\left(4\sqrt{h}\right)\left(4h+S\right)}-\frac{1}{8h\sqrt{S}}\frac{e^{-4\sqrt{h}\rho}}{ch\left[2\sqrt{h}\rho+\ln\left(\frac{2\sqrt{h}}{\sqrt{S}}\right)\right]} \quad (57)$$

3. CONCLUSIONS

We presented an analytical formalism in order to describe propagation in quadratic nonlinear media of periodic waves in Kerr type saturable. We, first, solved the Schrödinger equation with photonic network used Laplace transformation. The propagation properties were found by using different forms of saturable nonlinearity. However, an exact analytic solution of the propagation problem presented here creates possibilities for further theoretical investigation. Summarizing, we have shown the existence of completely stable periodic wave patterns in both saturable Kerr-type and quadratic nonlinear media.

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