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REGARDING SOME FUNDAMENTAL ASPECTS OF DESIGN AND DEVELOPMENT OF BESSEL ULTRASONIC CONCENTRATORS USED IN NONCONVENTIONAL MACHINING

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ABSTRACT: In the category of nonconventional machining, the use of concentrated energy from ultrasonic mechanical oscillations has long been a practical solution for efficient technological utilization. This work comprehensively develops and addresses an algorithm for sizing circular cross-section ultrasonic concentrators determined by Bessel functions. Bessel functions are canonical solutions to the Bessel differential equation in cylindrical coordinates, as they appear in Laplace's equation or in spherical coordinates derived from Helmholtz's equation. In industrial applications, optimizing process parameters such as vibration amplitude, amplification factor, and mechanical stress in the concentrator is of interest.

KEYWORDS: ultrasonic concentrators, elastic systems, mechanical waves, Bessel functions

1. INTRODUCTION

Ultrasonics are mechanical vibrations of an elastic medium, typically with frequencies in the range of 20 kHz to 10 MHz. A local disturbance produced in an elastic medium propagates from point to point through the oscillations of particles, as a mechanical wave.

Ultrasonic waves propagate through material media. Ultrasonics used in high-energy industrial applications are characterized by resonance frequency, propagation velocity, acoustic impedance or coupling, and vibration amplitude. The energy of ultrasonics can be generated through various methods such as aero or hydrodynamic, ionic, electrodynamic, magnetostrictive, or piezoelectric effects.

From an industrial application perspective, ultrasonics find usage in various domains, including processing, cleaning, welding, additional activation of other processing and treatment procedures, similar to conventional processes like machining (turning, drilling, grinding, polishing, etc.), plastic deformation (molding, forging, etc.), casting, as well as nonconventional processes like electro-erosion, electrochemistry, chemical, physical, and biological processes such as homogenization, sedimentation, particle deposition, with applications in the food industry, chemistry, medicine, etc.

From a structural-functional point of view, any mechanical equipment that utilizes ultrasonic energy features an electroacoustic transducer.

The electroacoustic transducer converts electrical energy at ultrasonic frequencies into mechanical energy of the same frequency but with different amplitude, which it localizes and focuses in the technological workspace. It consists of a unified assembly that includes a transducer, an ultrasonic concentrator, which can also be called a diffuser, sonotrode, vibrating membrane, etc., depending on the application, and the tool (where applicable).

Ultrasonic concentrators amplify or diminish the vibration amplitude of the transducer. They are made of steel, aluminium alloys, titanium alloys, etc. [1, 2] and can have different geometrical configurations depending on the technological application. Thus, ultrasonic concentrators can have a simple geometric configuration (cylindrical in straight lines, conical, exponential, catenoidal) or a complex one. Those with complex shapes are practically made up of several sections of simple concentrators.

Bessel became known for the class of functions that bear his name. Bessel introduced these functions in analysis, and they currently have significant importance in physics, engineering, and astronomy. Bessel functions are related to the problem of harmonic motions, have applications in materials science, in the study of circuits used in radio frequency technology.

In the field of nonconventional technologies, particularly those related to ultrasonic processing, the importance of Bessel functions arises from their ability to solve problems related to static potential and wave propagation. This includes electromagnetic waves in cylindrical waveguides, the vibration modes of a thin circular membrane used in sound signal reproduction (loudspeakers), solving problems of aeroelastic divergence, etc. Despite the diversity of applications, Bessel functions have not been extensively studied in the field of ultrasonics. This is why the purpose of this work is to identify a calculation algorithm for the construction of ultrasonic concentrators of circular section and profile based on Bessel functions used in nonconventional processing.

2. TRANSMITTING ULTRASONIC WAVES THROUGH ANY ARBITRARY MATERIAL MEDIUM

Consider a given material system (device, machine tool, concentrator, etc.) in a state of rest or permanent motion, also referred to as reference states. In some situations, the material system may undergo motion relative to these reference states, which can be studied with a certain number of parameters. These states are called vibrations or oscillations if the parameters describing the motion vary periodically over time around values corresponding to the reference states.

For the propagation of elastic vibrations, i.e., periodic variations in the state of the medium, the presence of a material system, known as a resonant acoustic system, is required. As the wave passes through the system, particles deviate from the reference state in a certain way. Elastic forces tend to return the particles to their initial position, while inertial forces allow the particles to oscillate around a mean position after the wave has passed. According to Kazantsev and Rosenberg [3], the velocity of the moving particles determines the kinetic energy of the particle flux, while the interaction forces between particles determine the potential energy. The sum of these two energies, referred to as acoustic or sound energy after Bindal [4], propagates from the ultrasonic source and attenuates as it travels through the medium.

Ultrasonic waves passing through a material medium are subject to absorption, which is influenced by viscosity (internal friction forces), thermal conductivity, and medium absorption. The absorption of elastic vibrations in different materials follows an exponential law and intensifies when the size of the medium's particles becomes comparable to the wavelength of ultrasound, as given by the relationship:

$$\xi_x = \xi_0 . e^{-\alpha x} \qquad (1)$$

where: ξ_x is the vibration amplitude at a distance x from the source,
ξ_0 is the vibration amplitude at $x=0$,
α is the absorption coefficient.

When waves propagate through a material medium, they undergo reflections, refractions, diffractions, interferences, and other phenomena characteristic of wave motion. Acoustic waves can vary depending on the trajectory that the particles of the material system can have, the nature, and the size of the medium through which they propagate. If the trajectory is linear, and the particles' displacement occurs in the direction of wave propagation, they are called longitudinal waves. Longitudinal waves can propagate through any elastic medium, whether it is gaseous, liquid, or solid, and they carry energy along the direction of displacement. In industrial applications, other types of waves such as quasi-longitudinal, transverse, bending, surface, torsional, symmetric, or asymmetric plate waves, etc., can also be encountered, but only longitudinal waves are significant for optimizing the shapes of ultrasonic concentrators.

Consider an elastic system represented by a straight bar with a constant cross-sectional area S. Under longitudinal deformation, the normal sections along the longitudinal axis move parallel to themselves. The deformation pattern of the straight bar with a constant cross-section is illustrated in Figure 1.

The section x_1 at time $t = 0$ will occupy the position $x_1 + \xi_1$ at the immediate subsequent moment, $t = \tau_1$, due to the unit effort $p(x)_1$. Similarly, the section that occupies the position x_2 at time $t = 0$ will move to position $x_2 + \xi_2$ at time $t = \tau_2$ under the influence of the unit effort $p(x_2)$. The instantaneous position of a section is characterized by the complex function $\xi(x, t)$, which represents the displacement of the section with abscissa x at time t.

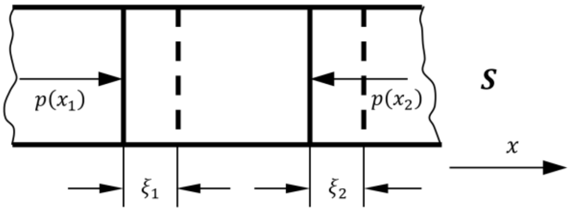

Figure 1. Longitudinal Deformation of a Straight Bar with Constant Cross-Section

The elastic binding forces arise from Hooke's law, which states that these forces are proportional to the deformation due to changes in the relative positions of two sections [5]:

$$p = -E\frac{\partial\xi}{\partial x} \tag{2}$$

where: p represents the unit stress,
E is the modulus of elasticity (Young's modulus).

The unit stress, considered as a compressive force, represents the force per unit area with which the elastic system, i.e., the rest of the bar, acts on a portion of the bar. The unit restoring force with which the bar acts on the exterior is equal in magnitude and opposite in direction to the unit stress.

The equation for longitudinal waves in bars with constant cross-section is derived from Newton's law for a section of the bar with thickness *dx* and specific mass *ρ*, assuming that the function *ξ(x,t)* is sufficiently smooth:

$$\rho\frac{\partial^2\xi}{\partial x^2}dx = p(x) - p(x+dx) = \\ = -E\left[\left.\frac{\partial\xi}{\partial x}\right|_x - \left.\frac{\partial\xi}{\partial x}\right|_{x+dx}\right] \tag{3}$$

This equation becomes:

$$\frac{\partial^2\xi}{\partial x^2} = \frac{1}{c^2}.\frac{\partial^2\xi}{\partial t^2} \tag{4}$$

where: $c = \sqrt{\frac{E}{\rho}}$ represents the speed of longitudinal wave propagation in a linear, isotropic, homogeneous, and conservative medium.

Equation (4) represents the equation of motion for the bar in the absence of losses (dissipation) and distributed forcing, also known as the wave propagation equation. Solutions to this equation are obtained by specifying boundary conditions. Stanomir [6], for example, sought solutions using the method of separation of variables, by assuming:

$$(x,t) = K.X(x).e^{st} \text{ and} = \alpha + i\omega \tag{5}$$

where: K is a material constant,
s is a complex number,
$X(x)$ is the amplitude of vibration.

Amza and colleagues [7] established that the solution to equation (4) should have the form:

$$\xi(x,t) = \psi_1(x-ct) + \psi_2(x+ct) \tag{6}$$

where: ψ_1 and ψ_2 are arbitrary functions.

In the case of bars with variable cross-section and distributed mass along their length, inertia forces cause a certain distribution of displacements and mechanical stresses over time t and coordinate x during vibration. The general equation that characterizes the propagation of longitudinal waves in bars with variable cross-section is given by Merkulov [8] and confirmed by Popescu [9], and it has the expression:

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\mu}{E}.\frac{\partial^3\xi}{\partial t\partial x^2} + \frac{1}{S}.\frac{\partial S}{\partial x}\left(\frac{\partial\xi}{\partial x} + \frac{\mu}{E}.\frac{\partial^2\xi}{\partial t\partial x}\right) = \\ = \frac{1}{c^2}.\frac{\partial^2\xi}{\partial t^2} \tag{7}$$

where: S represents the variable cross-sectional area with respect to the coordinate x according to a certain law,
μ is the coefficient of internal friction.

In the hypothesis that the material system (bar with variable cross-section) is perfectly elastic and non-dissipative, meaning μ=0 and oscillates according to a certain law, the differential equation (7) becomes:

$$\frac{\partial^2\xi}{\partial x^2} + \frac{1}{S}\frac{\partial S}{\partial x}\frac{\partial\xi}{\partial x} = \frac{1}{c^2}.\frac{\partial^2\xi}{\partial t^2} \tag{8}$$

In the case of harmonic oscillations of a bar with a variable cross-section, generated according to a law of the form:

$$\xi(x,t) = \xi(x).sin(\omega t + \varphi) \tag{9}$$

Equation (8) will be described by a second-order differential equation in terms of a single coordinate:

$$\frac{\partial^2\xi}{\partial x^2} + \frac{1}{S}.\frac{\partial S}{\partial x}.\frac{\partial\xi}{\partial x} + \frac{\omega^2}{c^2(1+j\eta)}.\xi = 0 \tag{10}$$

where: $\omega = 2\pi f$ represents the angular frequency of harmonic oscillations,
f is the frequency of vibrations,
$\eta = \mu\frac{\omega}{E}$ is the loss coefficient.

In the case of bars with variable cross-section and distributed mass made of steel or aluminium alloys with titanium, which are commonly used in machine construction, $\eta = 1.4 - 5.0x10^{-4}$, and it can be neglected. Noting $k = \frac{\omega}{c} = \frac{2\pi f}{\lambda f} = \frac{2\pi}{\lambda}$ as the wave number, where λ is the wavelength, equation (10) becomes:

$$\frac{\partial^2\xi}{\partial x^2} + \frac{1}{S}.\frac{\partial S}{\partial x}.\frac{\partial\xi}{\partial x} + k^2.\xi = 0 \tag{11}$$

Applying the law of conservation of momentum and mass under the acceptance of several simplifying assumptions as per Luca and Stan [10] (plane section hypothesis, absence of parasitic transverse oscillations, longitudinal oscillations following a harmonic law, homogeneous material of the concentrator, etc.), the general equation (7) becomes:

$$\frac{\partial^2\xi}{\partial x^2} + \frac{1}{S}\frac{\partial S}{\partial x}\frac{\partial\xi}{\partial x} + k^2 v = 0 \tag{12}$$

where: $v = \frac{\partial \xi}{\partial t}$ represents the velocity of oscillation of the particles in the elastic medium.

The wave propagation equation in bars with variable cross-section (equation 11) is known in the specialized literature as Webster's equation. However, it is derived from the theory of sonics developed by Gogu Costantinescu in liquid media [11]. Under given initial conditions (the profile of the bar along the generator, the continuity assumption in the separation plane, etc.), equations (11) or (12) allow for the determination of acoustic parameters (vibration amplitude, oscillation velocity, amplification factor, tensions in separation planes, etc.) that are specific to ultrasonic concentrators.

3. OPTIMIZATION OF THE CONSTRUCTION OF ULTRASONIC CONCENTRATORS WITH CIRCULAR CROSS-SECTION AND PROFILES BASED ON BESSEL FUNCTIONS

In mathematics, Bessel functions refer to the canonical solutions $\xi(x)$ of the differential equation:

$$x^2 . \frac{d^2\xi}{dx^2} + x . \frac{d\xi}{dx} + (x^2 - \nu^2) . \xi = 0 \qquad (13)$$

where: x represents the real variable (the coordinate of the point along the 0x direction),
ν is a complex parameter.

According to Watson [12], by applying the method of separation of variables to solve the Laplace equation and the Helmholtz equation in cylindrical or spherical coordinates, the Bessel equation is derived from which Bessel functions are obtained.

The optimization scheme for ultrasonic concentrators with a circular cross-section and a profile based on the Bessel function was developed by Nani and Horak [13] and is presented in Figure 2.

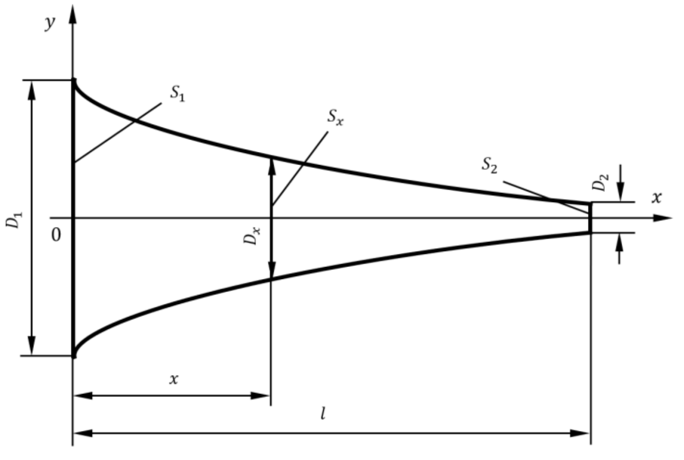

Figure 2. Diagram of an ultrasonic concentrator with a circular cross-section and profile based on Bessel functions.

The input diameter *D₁* of section *S1* is correlated with the transducer's cross-section, while the output diameter *D₂* with section *S₂* corresponds to the technological application characterized by the required frequency and vibration amplitude. The length of the ultrasonic concentrator is $l = \lambda/2$, and D_x corresponds to the diameter of section S_x located at a distance x from the separation plane with the transducer.

The elastic binding forces arise from Hooke's law, which states that these forces are proportional to the deformation due to changes in the relative positions of two sections [5]:

$$\begin{aligned} \xi(x) &= A . J_\nu(x) + B . Y_\nu(x) = \\ &= A' . H_\nu^{(1)}(x) + B' . H_\nu^{(2)}(x) \end{aligned} \qquad (14)$$

where: $J_\nu(x)$ represents the Bessel function of the first kind and order ν,
$Y_\nu(x)$ represents the Bessel function of the second kind and order ν,
$H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are the Bessel functions of the third kind and order ν,
A, B, A' and B' are constants determined from boundary conditions.

The four Bessel functions have the following properties:

a) $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent if ν is not an integer.

b) $J_n(x) = (-1)^n . J_{-n}(x)$ for integer *n*;

c) $Y_\nu(x) = \frac{1}{\sin(\nu\pi)} . [J_\nu(x) . \cos(\nu\pi) - J_{-\nu}(x)]$ if ν is not an integer; otherwise, take the limit of the fraction.

d) $J_\nu(x)$, $Y_\nu(x)$ and $H_\nu^{(1)}$, $H_\nu^{(2)}$ are pairwise linearly independent.

e) $H_\nu^{(1)}$ and $H_\nu^{(2)}$ can be determined as $H_\nu^{(1)} = J_\nu(x) + J . Y_\nu(x)$, and $H_\nu^{(2)} = J_\nu(x) - J . Y_\nu(x)$.

The function $J_n(x)$ has a series expansion:

$$J_n(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} . \frac{(-1)^k}{k!(n+k)!} \qquad (15)$$

For a fixed n and $|x| < 1$, the expansions hold [12, 14, 15]:

$J_0(x) = 1 - \frac{x^2}{4}$, Bessel function of the first kind and order *0*;
$Y_0(x) \approx \frac{2}{\pi} . \ln x$, Bessel function of the second kind and order *0*.

By substituting these functions into the differential equation (13), you obtain:

$$\xi(x) = A.J_0(x) + B.Y_0(x) \tag{16}$$

which means $\xi(x) = A.\left(1 - \frac{x^2}{4}\right) + B.\frac{2}{\pi}.lnx$, and the constants A and B are determined by imposing boundary conditions. Since the Bessel function of the second kind $Y_0(x)$ becomes infinite at $x = 0$, a particular solution of equation (16) is adopted, which becomes:

$$\xi(x) = A.J_0(x) - \frac{B}{k^2.F} \tag{17}$$

where: $F = E.S(x).{}^{d\xi}/_{dx}$, represents the elastic force between the particles of the ultrasonic concentrator,
E is Young's modulus.

If $B = 0$, then the oscillations in the concentrator are free, and for $B \neq 0$, the oscillations are forced (sinusoidal steady state), which is a characteristic situation for transmitting acoustic energy through the concentrator from the transducer to the tool or directly to the working medium.

For $x = 0$, you have $\xi(0) = \xi_1$, and equation (17) becomes in this case:

$$\xi_1 = A - \frac{B}{k^2.F} \tag{18}$$

By imposing the condition ${}^{d\xi}/_{dx} = 0$ for $x = l$ in equation (16), you obtain $B = A.\frac{\pi l^2}{4}$. Substituting this into equation (18) leads to:

$$\xi_1 = A.\left(1 - \frac{\pi l^2}{4k^2.F}\right) \tag{19}$$

The vibration amplitude at the exit of the concentrator, ξ_2, is determined from equation (16) by setting $x = l$:

$$\xi_2 = A.\left(1 - \frac{l^2}{4} + \frac{l^2}{2}.lnl\right) \tag{20}$$

From equations (19) and (20), we can deduce the amplification factor of the vibration amplitude:

$$k_A = \frac{\xi_2}{\xi_1} = \frac{k^2F.(4 - l^2 + 2l^2.lnl)}{4k^2F - \pi l^2} \tag{21}$$

where: K is a material constant,
s is a complex number,
$X(x)$ is the amplitude of vibration.

From equations (19) and (20), we can deduce the amplification factor of the vibration amplitude:

$$S=mc \tag{2}$$

Under given technological conditions, *D*₁, *S*₁, and the vibration amplitude $\xi(0) = \xi_1$ imposed by the shape and construction of the transducer, the resonance frequency of the elastic system *f*, the required vibration amplitude ξ_2 for the technological process, and the power of the ultrasonic generator are typically known. Through calculations, constructive elements of the ultrasonic concentrator, such as *D*₂, *S*₂, the position of the node of oscillation x, mechanical tension ${}^{d\xi}/_{dx}$, and the amplification factor k_A can be determined. The profile of ultrasonic concentrators with circular cross-sections based on Bessel functions of the first kind $J_0(x)$ and the second kind $Y_0(x)$ and order *0* is illustrated in Figure 3.

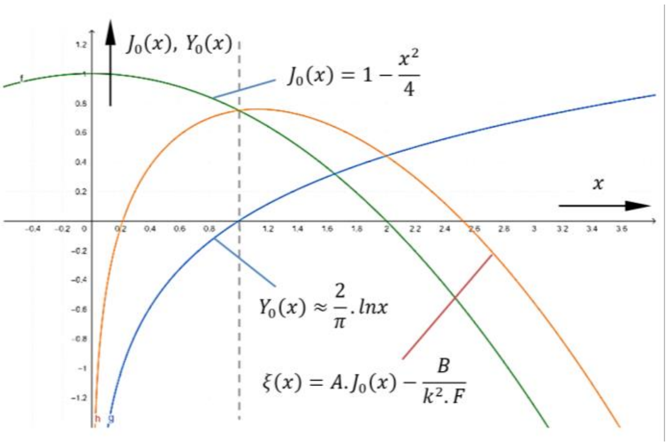

Figure 3. Variation of the profile of ultrasonic concentrators with circular cross-sections based on Bessel functions of the first and second kind and order *0*

It can be observed that for *n=0* and the random variable |*x*|<*1*, the profile of the ultrasonic concentrator is a combination of a parabolic and exponential profile.

Expanding equation (11) in dimensionless variables, we obtain:

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\partial}{\partial x}\ln(S(x)).\frac{\partial\xi}{\partial x} + \lambda^2.\xi = 0 \tag{22}$$

For the optimization of the profile according to equation (22), Eller's method [16] is used, which is based on approximating the functions $\xi(x)$, ${}^{\partial\xi}/_{\partial x}$, and ${}^{\partial^2\xi}/_{\partial x^2}$ with polynomials whose coefficients are determined from boundary conditions imposed by the specific technological application.

The values of the section $S(x)$ for designing the ultrasound concentrator profile based on Bessel functions in the domain of the random variable $x \in [0, 1]$, are further determined using the relationship [16]:

$$S(x) = S_1.e^{-f(x)} \tag{23}$$

where: $f(x)$ is the function associated with the dimensionless equation (22) and is given by:

$$f(x) = \int_0^x \frac{\frac{\partial^2\xi}{\partial x^2} + \lambda^2.\xi}{\frac{\partial\xi}{\partial x}} dx \tag{24}$$

4. CONCLUSION

Specialized literature has analysed, developed, and produced a multitude of ultrasound concentrators, ranging from simple to complex, with or without attached tools. The general aspects described and analysed above address, for the first time, the opportunity to optimize the shape of ultrasound concentrators with a profile generated by Bessel functions. Compared to known types of concentrators, the Bessel concentrator ensures amplification and vibration amplitude while minimizing mechanical stresses, thus ensuring increased durability of the electroacoustic transducer.

Although the calculations involved in determining the roots of Bessel functions and the constants A and B may seem complex and challenging at first glance, these obstacles can be easily overcome today thanks to specialized software and suitable manufacturing and control equipment.

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